

# Regular Behaviours with Names

## On the Rational Fixpoint of Endofunctors on Nominal Sets

Stefan Milius, Lutz Schröder, **Thorsten Wißmann**



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Automata,  
Trees, ...

involving

Freshness,  
Binding,...

Coalgebras

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State-finite

Automata,  
Trees, ...

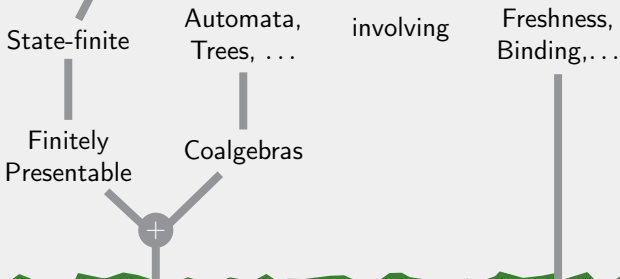
involving

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On the Rational Fixpoint of Endofunctors on Nominal Sets

# Nom: The Category of Nominal Sets

## Nominal Set $X$

A set  $X$  with

- Group action  $\mathfrak{S}_f(\mathcal{V}) \times X \rightarrow X$

Renaming atoms



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Renaming atoms

Visible atoms

$\lambda$ -terms

$\text{supp}(x) =$  all variables in  $x$

$\lambda$ -terms mod  $\alpha$ -equivalence

$\text{supp}(x) =$  **free** variables in  $x$

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## Equivariants $f : X \rightarrow Y$

Maps commuting with group actions:  $\pi \cdot f(x) = f(\pi \cdot x)$ .

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Gabbay, Pitts'99

$LX = \mathcal{V} + \mathcal{V} \times X + X \times X$

$\xrightarrow{q_X}$

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## Final $L$ -Coalgebra

$\lambda$ -trees involving finitely many variables

$\dashrightarrow$

## $L_\alpha$ -Coalgebras

$\lambda$ -trees modulo  $\alpha$ -equivalence involving finitely many free variables

Kurz, Petrisan, Severi, de Vries'13

# The Rational Fixpoint: Regular Behaviours

$\mathcal{C}$  lfp category,  $F$  finitary

$$\text{f.p. } \mathcal{C} \xrightarrow{c} FC$$

Adámek, Milius, Velebil'06

# The Rational Fixpoint: Regular Behaviours

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$\mathcal{C} = \text{Set}$

$FX = 2 \times X^A$

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finite

finite automata

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$D \longrightarrow FD$

locally finite

locally finite  
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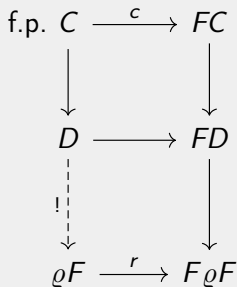
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Rational Fixpoint

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locally finite

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final  
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Regular  
Languages

Adámek, Milius, Velebil'06

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 \vdots & & \downarrow \\
 \varrho F & \xrightarrow{r} & F\varrho F
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Rational Fixpoint =  $\text{colim}_{\mathcal{C} \text{ f.p.}} C \xrightarrow{c} FC$

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# Finitely Presentable Objects in Nom

orbit-finite = finitely presentable in Nom

$x, y$  in the same orbit  $\Leftrightarrow \pi \cdot x = y$  for some  $\pi$ .

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Either singleton or **infinite**

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Behaviour of  $C \rightarrow FC$ ,  $C$  orbit-finite?

?

# Regular Behaviours with Names

Rational  $\lambda$ -trees =  $\rho L$

$$LX = \mathcal{V} + \mathcal{V} \times X + X \times X$$

Rational  $\lambda$ -trees mod  $\alpha$  =  $\rho L_\alpha$

$$L_\alpha X = \mathcal{V} + [\mathcal{V}]X + X \times X$$

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$$DX = 2 \times X^\mathcal{V}, \quad KX = 2 \times X^\mathcal{V} \times [\mathcal{V}]X,$$

Functors from Binding Signatures, ...

?

# Regular Behaviours with Names

$$\text{Nom} \xrightarrow{\bar{F}} \text{Nom}$$

$$\begin{array}{ccc} \text{U} \downarrow & & \downarrow \text{U} \\ \text{Set} & \xrightarrow{F} & \text{Set} \end{array}$$

$$\text{Set} \xrightarrow{F} \text{Set}$$

Lifting of a Set-Function

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Lifting of a Set-Functor

$$G \xrightarrow{q} H$$

Quotient of another Nom-Functor

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# Part 1: Liftings

# Liftings: Assumptions

Assumption:  $\bar{F} : \text{Nom}^{\downarrow}$  a localizable lifting, i.e.

- 1  $\bar{F}$  comes from a distributive law  $\lambda$  over  $F = U\bar{F}D$ .
- 2 For each  $W \subseteq \mathcal{V}$ ,  $\lambda$  restricts to:  
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## Examples

- Constants, Identity, Finitary Powerset  $\mathcal{P}_f$
- Closed under finite products, coproducts, composition.

# Locally Finite vs Locally Orbit-Finite

$c : C \rightarrow FC$  in Set vs  $c : C \rightarrow \bar{F}C$  in Nom  
 $C$  finite  $\not\iff$   $C$  orbit-finite

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For  $c : C \rightarrow \bar{F}C$ ,  $c$  locally finite in Set  $\implies$   $c$  is locally orbit-finite

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$(\varrho F, r) = \text{Rational Fixpoint of } \bar{F} : \text{Nom}$

Group action by corecursion. Finite support by coinduction.

## Part 2: Quotients of Nom-functors

# Quotients of coalgebras

$$G : \text{Nom}^{\downarrow} \longrightarrow H : \text{Nom}^{\downarrow}$$

$$\varrho G \xrightarrow{r^G} G \varrho G \xrightarrow{q_{\varrho G}} H \varrho G$$

$$\varrho H \xrightarrow{r^H} H \varrho H$$

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 \text{surjective?} \rightarrow h \downarrow & & & & \downarrow Hh \\
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 \end{array}$$

# Constructing backwards along the quotient

**Definition**  $X < Y := \{(x, y) \in X \times Y \mid \text{supp}(x) \subseteq \text{supp}(y)\}$

Sub-strength  $s_{X,Y} : GX < Y \rightarrow G(X < Y)$ ,

not necessarily natural, but

$$\begin{array}{ccc}
 GX < Y & \xrightarrow{s_{X,Y}} & G(X < Y) \\
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## Theorem

Finitary  $G : \text{Nom}^{\circlearrowleft}$  with sub-strength &  $q : G \twoheadrightarrow H$ , then  $qG \twoheadrightarrow qH$

# Applications

$$F : \text{Set}^{\curvearrowright} \rightsquigarrow \bar{F} = G : \text{Nom}^{\curvearrowright} \rightarrow H : \text{Nom}^{\curvearrowright}$$

# Example: Binding

$$LX = \mathcal{V} + \mathcal{V} \times X + X \times X \xrightarrow{q} L_\alpha X = \mathcal{V} + [\mathcal{V}]X + X \times X$$

Generally:  $L_\alpha =$  Arbitrary Binding Signature,  $L =$  Polynomial

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Kurz, Petrisan, Severi, de Vries'13

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# Example: Exponentiation by Atoms $X^{\mathcal{V}}$

$$FX = \mathcal{V} \times X \times \prod_{n \in \mathbb{N}} (\mathcal{V} \times X)^n \xrightarrow{q} HX = X^{\mathcal{V}}$$

default case                      exceptions

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default case
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## Various Kinds of Nominal Automata

- $FX = 2 \times X^{\mathcal{V}}$
- $KX = 2 \times X^{\mathcal{V}} \times [\mathcal{V}]X$
- $NX = 2 \times \mathcal{P}_f(X^{\mathcal{V}}) \times \mathcal{P}_f([\mathcal{V}]X)$

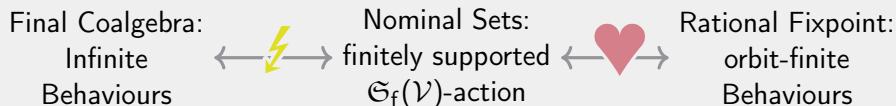
# Open Questions

?

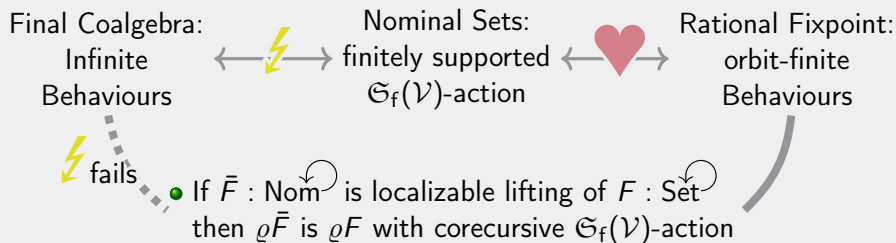
More applications?  
Necessity of assumptions?  
Any non-examples beside technical ones?

?

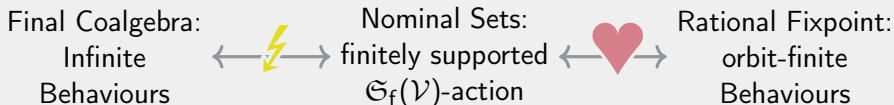
# Main Results



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- If  $\bar{F} : \text{Nom}^{\curvearrowright}$  is localizable lifting of  $F : \text{Set}^{\curvearrowright}$  then  $\varrho\bar{F}$  is  $\varrho F$  with corecursive  $\mathfrak{S}_f(\mathcal{V})$ -action
- If  $G : \text{Nom}^{\curvearrowright}$  is a quotient  $H : \text{Nom}^{\curvearrowright}$  with a substrength then  $\varrho G$  is a quotient of  $\varrho H$

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# Assumption 1: Comming from a Distributive Law

$\mathfrak{S}_f(\mathcal{V})$ -action on a Nominal Set  $X$

Eilenberg-Moore algebra on  $X$  for the monad  $TY = \mathfrak{S}_f(\mathcal{V}) \times Y$

| Liftings   | $\iff$ | Distributive Laws   |
|--|--------|---|
| $\begin{array}{ccc} \text{Set}^T & \xrightarrow{\bar{F}} & \text{Set}^T \\ U \downarrow & & \downarrow U \\ \text{Set} & \xrightarrow{F} & \text{Set} \end{array}$ | $\iff$ | $\lambda : TF \rightarrow FT$ <p>preserving<br/>monad structure</p> |

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 U \downarrow & \iff & \text{preserving} \\
 \text{Set} \xrightarrow{F} \text{Set} & & \text{monad structure}
 \end{array}$$

Assumption

$\bar{F} : \text{Nom} \rightarrow \text{Nom}$  comes from a Distributive Law  $\mathfrak{S}_f(\mathcal{V}) \times \_$  over  $F$   
 Mapping nominal sets to nominal sets.

## Assumption 2: Localizability

$\lambda : \mathfrak{G}_f(\mathcal{V}) \times F\_ \rightarrow F(\mathfrak{G}_f(\mathcal{V}) \times \_)$  localizable

For each  $W \subseteq \mathcal{V}$ ,  $\lambda$  restricts to  $\mathfrak{G}_f(W)$ :

$$\begin{array}{ccc}
 \mathfrak{G}_f(\mathcal{V}) \times FX & \xrightarrow{\lambda_X} & F(\mathfrak{G}_f(\mathcal{V}) \times X) \\
 \uparrow & & \uparrow \\
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 \end{array}$$

### Non-Example

For  $F = \text{Id}_{\text{Set}}$ ,  $\lambda(\pi, x) = (g \cdot \pi \cdot g^{-1})$ ,  $g$  fixed.

# Something like “projective objects” in Nom

## Definition: strongly supported

Some  $x \in X$  is **strongly supported** iff

$$\pi \cdot x = x \implies \forall v \in \text{supp}(x) : \pi(v) = v$$

## Examples

$\mathcal{V}^n$  is strongly supported.  $\mathcal{P}_f(\mathcal{V})$  not.

**Proposition** (Mentioned already in Kurz, Petrisan, Velebil’10)

$X, Y$  nominal sets,  $X$  strongly supported,  $O \subseteq X$  a choice of one element from each orbit. Then any map  $f_0 : O \rightarrow Y$  with

$$\text{supp}(f_0(x)) \subseteq \text{supp}(x)$$

extends uniquely to an equivariant  $f : X \rightarrow Y$ .

# Applied to our $C < W$

## Lemma

There is a strong  $W$  and an equivariant map  $f : C < W \rightarrow GC$  such that:

$$\begin{array}{ccc}
 C < W & \xrightarrow{f} & GC \\
 \text{outl} \downarrow & & \downarrow q_C \\
 C & \xrightarrow{c} & HC
 \end{array}$$

# Applied to our $C < W$

## Lemma

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## Proposition

$c : C \rightarrow HC$  is via  $\text{outl}$  a quotient of the orbit-finite

$$C < W \xrightarrow{\bar{f}} GC < W \xrightarrow{s_{C,W}} G(C < W)$$

(where  $\bar{f}(x, w) = (f(x), w)$ ).